Stress Distribution and Displacements in a Lubricated Elastic Seal

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Theme

THE static displacements and stresses in a lubricated elastic pressurized seal are studied by the complex variable method of plane stress.

Contents

To prevent the escape of pressurizing gases from rotary or actuator shafts, the elastic seal between the shaft and housing must be designed to withstand the pressure forces acting on it. Since theoretical solutions of stresses and displacements are useful in designing such seals, we consider the problem of the seal on a shaft against a fixed housing as shown in the cross section of Fig. 1. On the solid boundary surfaces, as well as on moving shafts, a lubricant reduces leakage between seal and housing as well as friction between seal and moving shafts. Accordingly, we assume that all surfaces of the seal are lubricated and hence, all surface shear forces vanish when the shaft is not being actuated. In addition, we have uniform pressure p on each segment of the free boundary. We assume that the shaft diameter is sufficiently large that plane elasticity describes adequately the stresses and displacements in the seal. The condition that displacements normal to the housing vanish results in an integral equation for the normal stress distributed along its surface.

To solve the boundary value problem, we apply the method of complex variables in the theory of plane elasticity as described by Milne-Thomson.[1] Let $F(z,\bar{z}) = 0$ denote a cylindrical surface in the $z = x + iy$ plane, x and y being the cartesian coordinates. Then from Eq. (7), p. 31, of Ref. 1, the normal stress $\widehat{nn}$, measured positive outward, and the shear stress $\widehat{ns}$ on this surface $F(z,\bar{z}) = 0$ are given by

$$2(\widehat{nn} - i\widehat{ns}) = w(z) + \bar{w}(\bar{z}) + [\bar{z}w'(z) + \omega(z)](dz/d\bar{z}) \quad (1)$$

where

$$dz/d\bar{z} = -(\partial F/\partial \bar{z})/(\partial F/\partial z)$$

Similarly, the displacements u and v in the x and y direction, respectively, are given by Eq. (9), p. 32, of Ref. 1 as

$$4\mu(u + iv) = k\int w(z)dz - z\bar{w}(\bar{z}) - \int \bar{\omega}(\bar{z})d\bar{z} \quad (2)$$

where $\mu = E/2(1+\eta)$ is the rigidity, $k = 3 - 4\eta$, E is Young's modulus, and η is Poisson's ratio. The function $w(z)$ is related to the stress boundary condition from the theorem of Plemelj by the integral over the surface

$$w(z) = \frac{1}{\pi i}\int_C \frac{(\widehat{nn} + i\widehat{ns})dt}{t - z} \quad (3)$$

We express the function $\omega(z)$ in terms of $w(z)$. The boundary equation, $F(z,\bar{z}) = 0$, may be solved for z in terms of $\bar{z}$ in the form $z = z(\bar{z})$. The point $z_r = z(\bar{z})$ lies outside of the boundary if z is in the region of the elastic material. Then for z inside the seal, $\omega(z)$ is given by Eq. (7), p. 111 in Ref. 1. Thus, we have

$$\bar{\omega}(\bar{z}) = [\bar{w}(\bar{z}) - w(z_r)]/\phi(z_r,\bar{z}) - z_r\bar{w}'(\bar{z}) \quad (4)$$

where

$$\phi(z,\bar{z}) = -(\partial F/\partial z)/(\partial F/\partial \bar{z})$$

To satisfy zero shear and zero lateral displacement on the shaft of Fig. 1a, we add the mirror image of the seal for x negative thus making the shaft a surface of symmetry of the boundary stresses. The stress boundary conditions then are, on the aperture with pressure p_1

$$\widehat{nn} = -p_1 \text{ for } -a \le x \le a,$$

and on the housing

$$\widehat{nn} = -N(|x|) \text{ for } a < |x| < b,$$

and $\widehat{nn} = p_2$ over the remaining portions of the boundary. The integral of Eq. (3) leads to

$$w(z) = -\frac{p_1}{\pi i}\log\left(\frac{z-a}{z+a}\right) - \frac{1}{\pi i}\int_a^b N(t)\left[\frac{1}{t-z} - \frac{1}{t+z}\right]dt - \frac{p_2}{\pi i}[g_1(z) + g_2(z) + g_3(z)] \quad (5)$$

where

$$g_1(z) = \log\left(\frac{z-b-ci}{z-b}\right), \quad \left\{\begin{matrix} -\pi/2 \le \arg(z-b-ci) \le 3\pi/2 \\ -\pi/2 \le \arg(z-b) \le 3\pi/2 \end{matrix}\right\}$$

$$g_2(z) = \log\left(\frac{z+b-ci}{z-b-ci}\right), \quad \left\{\begin{matrix} -\pi \le \arg(z+b-ci) \le \pi \\ -\pi \le \arg(z-b-i) \le \pi \end{matrix}\right\} \quad (6)$$

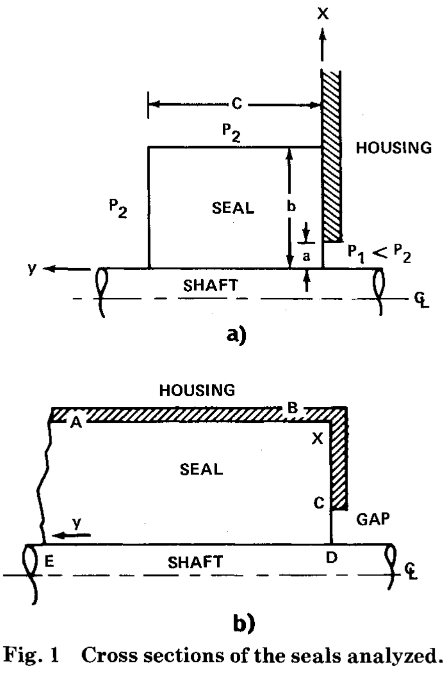

Fig. 1 Cross sections of the seals analyzed.

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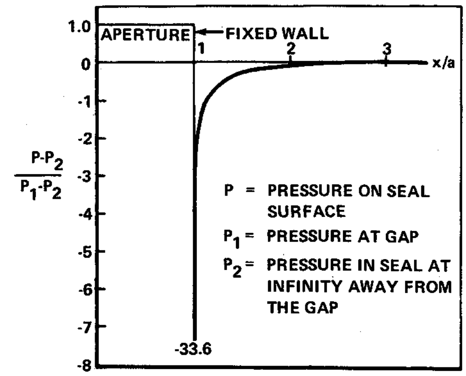

Fig. 2 Pressure distribution on the wall resulting from the solution of integral Eq. (7).

and

$$g_3(z) = \log\left(\frac{z+b}{z+b-ci}\right), \quad \begin{Bmatrix} -\pi \le \arg(z+b) \le 3\pi/2 \\ -\pi/2 \le \arg(z+b-i) \le 3\pi/2 \end{Bmatrix}$$

The stress distribution $N(x)$ is determined by setting the displacement normal to the wall, $y = 0$ and $a \le x \le b$, from Eq. (2) equal to zero. Application of Eqs. (3–5) to Eq. (2) ultimately leads to the following integral equation

$$\int_{a/b}^{1} g(\tau)\log|\tau^2 - s^2|d\tau - 2\int_{a/b}^{1} g(\tau)\log\tau d\tau = (1/s - 1)\log(1-s) - (1/s+1)\log(1+s) + 2 \tag{7}$$

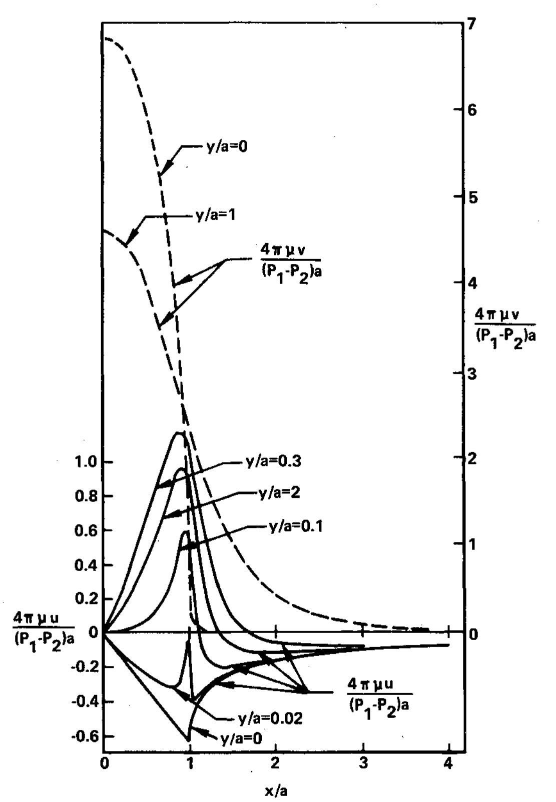

Fig. 3 Displacements *u* and *v*, normal and parallel, respectively, to the shaft, for the infinite seal of Fig. 1a.

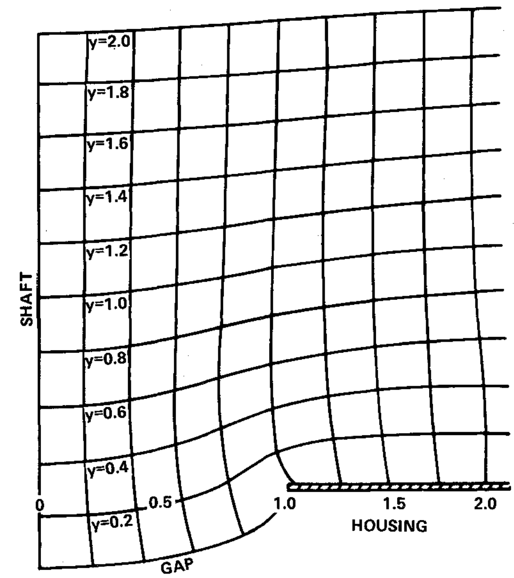

Fig. 4 Distortion of a rectangular net in the infinite seal of Fig. 1a.

together with the condition of static equilibrium

$$\int_{1}^{b/a} f(t)dt = -1 \tag{8}$$

Here $s = a/x$. The function $g(t)$ is related to the stress distribution $N(x)$ by

$$N(x) = p_2 + (p_1 - p_2)(a/x)^2 g(a/x) \tag{9}$$

The solution for the quarter space infinite seal in Fig. 1a ($a/b \to 0$) is given in Fig. 2. One interesting feature is the sharp pressure peak at the corner of the fixed wall. Away from the corner the pressure drops rapidly towards its value at infinity in two or three aperture lengths along the wall. Plots of the displacement in the y direction (normal to the wall) are shown as dashed lines in Fig. 3 for the seal surface $y = 0$ and for the distance of one gap length into the seal as functions of x. On the boundary of the seal $y/a = 0$ the displacement is a maximum at $x/a = 0$, the shaft end of the gap. The displacement in the x direction (along the wall) is shown for several values of y as the solid lines in Fig. 3. Along the aperture, the displacement in the x direction is linear in x and then goes rapidly to zero along the wall. One interesting aspect is the reversing of the direction of the peak x displacement for small values of y near the housing corner, $x/a = 1$. Also, as we move away from the aperture into the seal along the line $x/a = 1$, the x displacement reverses direction. These features are apparent in Fig. 4 which shows the distortion of a square grid on the seal resulting from the stress load.

The stress in the infinite seal of Fig. 1b, was solved by formulating it as a boundary value problem on the boundary plane of the elastic half space. The function $w(z)$ was derived and an integral equation obtained for the pressure distribution on the plate similar to Eq. (7) for the seal of Fig. 1a. However, no numerical results were obtained.

Reference

[1]Milne-Thomson, L. M., *Plane Elastic Systems,* Springer-Verlag, Berlin, West Germany, 1960,